



Research Article

Distribution of Concomitants from the Generalized Morgenstern Type Bivariate Exponential Distribution

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Received: 26/01/2026, Accepted: 18/04/2026, Published(online): 31/05/2026

Abstract

The generalized bivariate Morgenstern family extends the classical Morgenstern framework to enhance the dependence structure between variables. This study examines the dependence between the marginal variables through copula-based techniques and further explores the distributional properties of concomitants of order statistics arising from the generalized Morgenstern-type bivariate exponential distribution.

Keywords: Order statistics, Concomitants, Generalized Morgenstern, Copula, Bivariate exponential.

1 Introduction

Let $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$ be a random sample of size n drawn from a continuous bivariate population (X, Y) with c.d.f $F_{X,Y}(x, y)$. Arranging the sample in non-decreasing order of its X -component yields the corresponding order statistics $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$. The Y -values corresponding to these order

statistics are $Y_{[1:n]}, Y_{[2:n]}, \dots, Y_{[n:n]}$ and are referred to as the concomitants of order statistics. The concept of concomitants was introduced by David (1973) and independently by Bhattacharya (1974) under the term induced order statistics.

Extensive research has examined the distributional properties of concomitants, assuming i.i.d random variables. Foundational contributions include the works of David (1973), David and Galambos (1974), and Yang (1977). Subsequent notable studies were carried out by Bairamov et al. (2001), Scaria and Nair (2005, 2008), Beg and Ahsanullah (2008), Scaria and Thomas (2008), Ahsanullah (1994), Thomas and Veena (2014) and Thomas and Scaria (2011). Yeo and David (1984) demonstrated an important use of concomitants within selection procedures.

Scaria and Thomas (2013) introduced second order concomitants of order statistics to minimize measurements of highly expensive or difficult-to-measure objects. This technique has been further explored by Barakat et al. (2025), who studied second-order concomitants within the Cambanis family and examined related information measures.

When the process of data generation is governed by random mechanisms, it becomes necessary to determine appropriate probability distributions to represent the data precisely. In scenarios where two variables jointly contribute to the data generation, bivariate distributions provide the natural modelling framework. A well-known illustration is the Morgenstern family of distributions (1956), which is specified by the joint distribution function.

$$F_{X,Y}(x, y) = F_X(x)F_Y(y) \{1 + \alpha[1 - F_X(x)][1 - F_Y(y)]\}, \quad |\alpha| \leq 1. \quad (1)$$

The family of distributions defined by equation (1) constitutes a versatile bivariate modelling framework, particularly suited to contexts where initial information about the data is available in the form of marginal distributions. This system accommodates a wide spectrum of distributional shapes and ranges, making it applicable to diverse practical scenarios. In many instances, the joint distribution of (X, Y) becomes relevant when the marginal forms are known and the dependence structure, along with joint probabilities, is of analytical interest. For example, Johnson and Kotz (1975) introduced a bivariate chi-square model that naturally arises when the marginals follow chi-square distributions.

The family (1) serves as a suitable modelling framework in situations where the marginal distributions of the component variables are either explicitly known or can be reliably determined, and the dependence structure aligns with that of a

particular member of the system. Within this framework, notable investigations by D'Este (1981), Gumbel (1960, 1961), and Mardia (1970) have focused on members of family (1) whose marginals follow exponential, uniform, gamma and extreme value distributions.

An additional aspect of the Morgenstern family that has been examined in recent studies concerns the elementary characteristics of the concomitants associated with order statistics of the component variables. These properties provide a convenient framework for drawing inferences and formulating decision-making procedures. For comprehensive discussions, one may refer Bairamov et al. (2001), Balasubramanian and Beg (1997), Scaria and Nair (1999, 2005, 2008), as well as Nair and Scaria (2010). The analyses presented in these works highlight distinctive features of individual distributions, each offering valuable insights for data modelling and thereby enriching the systematic exploration of various families of the Morgenstern type.

To enhance the dependence structure between variables, various modifications of the bivariate Morgenstern distribution have been proposed in the literature. Notable modifications are by Huang and Kotz (1999), Bairamov and Kotz (2000) and Bairamov et al. (2001). Veena and Thomas (2008) examined a family $\mathcal{L}$ of bivariate distribution functions which may be regarded as an extension of the Morgenstern family of bivariate distributions defined in equation (1). The class $\mathcal{L}$ is constituted by all distribution functions that take the form given by

$$F_{X,Y}(x, y) = F_X(x)F_Y(y) + \sum_{i=1}^t \alpha_i \{F_X(x)[1 - F_X(x)]\}^{m_i} \{F_Y(y)[1 - F_Y(y)]\}^{p_i}. \quad (2)$$

where $F_X(x)$ and $F_Y(y)$ represent the marginal distributions of $F_{X,Y}(x, y)$. The constants m_i and p_i involved in (2) are real numbers greater than or equal to 1 and α_i are real constants confined to lie within an interval about zero, while t is assumed to be a positive integer. In particular when $t=1$

$$F_{X,Y}(x, y) = F_X(x)F_Y(y) + \alpha \{F_X(x)[1 - F_X(x)]\}^m \{F_Y(y)[1 - F_Y(y)]\}^p. \quad (3)$$

For $m = p = 1$, (3) simplifies the Morgenstern system defined in (1). The conditional distribution function of Y given $X=x$ is

$$\begin{aligned}
F_{Y|X}(y | x) &= \frac{\frac{\partial}{\partial x} F_{X,Y}(x, y)}{f_X(x)} \\
&= F_Y(y) + \alpha m \left\{ F_X(x) [1 - F_X(x)] \right\}^{m-1} \\
&\quad \times [1 - 2F_X(x)] \left\{ F_Y(y) [1 - F_Y(y)] \right\}^p. \quad (4)
\end{aligned}$$

In Section 2, we investigate the dependence structure of the generalized Morgenstern bivariate family of distributions, employing Kendall's tau and Spearman's rho. In Section 3, the distribution theory of concomitants of order statistics for the generalized Morgenstern-type bivariate exponential distribution is presented.

2 Dependence Measures

Dependence structures between random variables represent one of the most extensively investigated themes in distribution theory. Without explicit assumptions concerning such dependence, the construction of meaningful statistical models becomes infeasible. In this section, we therefore examine two dependence measures used to analyse the association between two random variables governed by the bivariate generalized Morgenstern distribution. Particular emphasis is placed on copulas, which play a central role in the study of dependence. Dependence can be quantified through various metrics, among which the most prominent scale-invariant measures are Kendall's tau and Spearman's rho. Both of these indices capture a specific aspect of dependence referred to as concordance. Two random variables are said to be concordant when higher values of one variable are typically associated with higher values of the other, and likewise, lower values of one correspond to lower values of the other.

As presented in Nelson (2006), Kendall's tau takes the form

$$\tau = 4 \int_0^1 \int_0^1 C(u, v) dC(u, v) - 1. \quad (5)$$

From (3)

$$C(u, v) = uv + \alpha [u^m(1 - u)^m v^p(1 - v)^p] \quad (6)$$

and

$$\begin{aligned}
 dC(u, v) &= \frac{\partial^2}{\partial u \partial v} C(u, v) \\
 &= \frac{\partial^2}{\partial u \partial v} [uv + \alpha (u^m (1-u)^m v^p (1-v)^p)] \\
 &= 1 + \alpha m p u^{m-1} (1-u)^{m-1} (1-2u) v^{p-1} (1-v)^{p-1} (1-2v). \quad (7)
 \end{aligned}$$

Thus, substituting (6) and (7) into (5), we obtain

$$\tau = 8\alpha m p \frac{\Gamma(m) \Gamma(m+1)}{\Gamma(2m+2)} \frac{\Gamma(p) \Gamma(p+1)}{\Gamma(2p+2)}. \quad (8)$$

Dependence strength is controlled by α, m and p . Positive α gives positive dependence, negative α gives negative dependence. The Gamma ratios act like weights that shrink or amplify dependence. Larger m or p generally reduce the value of τ , because the Gamma ratios decrease as the parameters grow. This means dependence weakens as the marginals become more “peaked” or heavy-tailed. Even with $\alpha = 1$, τ , does not reach ± 1 . This makes it suitable for modelling weak dependence structures.

Invoking the definition of Spearman’s rho, we arrive at

$$\rho = 12 \int_0^1 \int_0^1 [C(u, v) - uv] du dv. \quad (9)$$

Employing (6) within (9), we deduce that

$$\rho = 12\alpha B(m+1, m+1) B(p+1, p+1). \quad (10)$$

Just like Kendall’s tau, Spearman’s rho is proportional to α . For larger m or p , the Beta values in (10) decrease, so becomes smaller. This means dependence weakens as marginals become more concentrated or heavy-tailed. This makes it suitable for weak dependence modelling.

3 Distribution of Concomitants from Bivariate Generalized Morgenstern Type Exponential Distribution

Among the family introduced in (3), one notable member is the bivariate exponential distribution, defined through its distribution function,

$$F_{X,Y}(x, y) = \left[1 - \exp\left(\frac{-x}{\theta_1}\right)\right] \left[1 - \exp\left(\frac{-y}{\theta_2}\right)\right] + \alpha \left[\left(1 - \exp\left(\frac{-x}{\theta_1}\right)\right) \exp\left(\frac{-x}{\theta_1}\right)\right]^m \left[\left(1 - \exp\left(\frac{-y}{\theta_2}\right)\right) \exp\left(\frac{-y}{\theta_2}\right)\right]^p$$

$x, y > 0; \theta_1, \theta_2 > 0.$

From Thomas (2015), the c.d.f. of $Y_{[r:n]}$ is

$$F_{Y_{[r:n]}}(y) = F_Y(y) + \alpha m \frac{B(m+r-1, m+n-r)}{B(r, n-r+1)} \frac{n-2r+1}{2m+n-1} \{F_Y(y) [1 - F_Y(y)]\}^p.$$

(11)

From (11)

$$F_{Y_{[r:n]}}(y) = \left(1 - \exp\left(-\frac{y}{\theta_2}\right)\right) + \alpha m \cdot \frac{B(m+r-1, m+n-r)}{B(r, n-r+1)} \cdot \frac{n-2r+1}{2m+n-1} \times \left[\left(1 - \exp\left(-\frac{y}{\theta_2}\right)\right) \exp\left(-\frac{y}{\theta_2}\right)\right]^p, \quad 0 < y < \infty.$$

From Thomas (2015), p.d.f of $Y_{[r:n]}$ is

$$f_{Y_{[r:n]}}(y) = f_Y(y) \left[1 + \alpha m p \cdot \frac{B(m+r-1, m+n-r)}{B(r, n-r+1)} \cdot \frac{n-2r+1}{2m+n-1} \times \{F_Y(y) [1 - F_Y(y)]\}^{p-1} [1 - 2F_Y(y)]\right].$$

Hence

$$f_{Y_{[r:n]}}(y) = \frac{1}{\theta_2} \exp\left(-\frac{y}{\theta_2}\right) \left[1 + \alpha mp \cdot \frac{B(m+r-1, m+n-r)}{B(r, n-r+1)} \cdot \frac{n-2r+1}{2m+n-1} \right. \\ \left. \times \left[\left(1 - \exp\left(-\frac{y}{\theta_2}\right) \right) \exp\left(-\frac{y}{\theta_2}\right) \right]^{p-1} \left[2 \exp\left(-\frac{y}{\theta_2}\right) - 1 \right] \right], \quad y \geq 0. \quad (12)$$

Employing the formula for the density of order statistics in (12), we deduce

$$f_{Y_{[r:n]}}(y) = f_Y(y) + \alpha mp \cdot \frac{B(m+r-1, m+n-r)}{B(r, n-r+1)} \cdot \frac{n-2r+1}{2m+n-1} \cdot B(p, p+1) \\ \times [f_{p:2p}(y) - f_{p+1:2p}(y)] \quad (13)$$

$$= f_Y(y) + A[f_{p:2p}(y) - f_{p+1:2p}(y)] \quad (14)$$

where

$$A = \alpha mp \cdot \frac{B(m+r-1, m+n-r)}{B(r, n-r+1)} \cdot \frac{n-2r+1}{2m+n-1} \cdot B(p, p+1) \quad (15)$$

and $f_{r:s}(y)$ is the p.d.f of $Y_{r:s}$, $r, s = 1, 2, \dots$

From (14) the k^{th} moment of $Y_{[r:n]}$ may be obtained in the form

$$\mu_{[r:n]}(k) = \mu^{(k)} + A[\mu_{p:2p}^{(k)} - \mu_{p+1:2p}^{(k)}] \quad (16)$$

where

$$\mu^{(k)} = E(Y^k)$$

and

$$\mu_{r:s}^{(k)} = E(Y_{r:s}^k), \quad r : s = 1, 2, \dots$$

For an exponential distribution with mean θ , the expectation of the r^{th} order statistic from a sample of size n is expressed as

$$\mu_{r:n} = \theta \sum_{j=n-r+1}^n \frac{1}{j}.$$

Therefore, if $X_{p:2p}$ and $X_{p+1:2p}$ are the p^{th} and $(p+1)^{th}$ order statistics from a sample of size $2p$ from an exponential distribution with mean θ , then

$$\mu_{p:2p} = \theta \sum_{j=p+1}^{2p} \frac{1}{j}$$

and

$$\mu_{p+1:2p} = \theta \sum_{j=p}^{2p} \frac{1}{j}.$$

Hence

$$\begin{aligned} \mu_{p:2p} - \mu_{p+1:2p} &= \theta \left(\sum_{j=p+1}^{2p} \frac{1}{j} - \sum_{j=p}^{2p} \frac{1}{j} \right) \\ &= -\frac{\theta}{p}. \end{aligned} \tag{17}$$

Also

$$V(X_{p:2p}) = \theta^2 \sum_{j=p+1}^{2p} \frac{1}{j^2}$$

and

$$V(X_{p+1:2p}) = \theta^2 \sum_{j=p}^{2p} \frac{1}{j^2}.$$

Hence

$$\mu_{p:2p}^2 = \theta^2 \left[\left(\sum_{j=p+1}^{2p} \frac{1}{j} \right)^2 + \sum_{j=p+1}^{2p} \frac{1}{j^2} \right]$$

and

$$\mu_{p+1:2p}^2 = \theta^2 \left[\left(\sum_{j=p}^{2p} \frac{1}{j} \right)^2 + \sum_{j=p}^{2p} \frac{1}{j^2} \right].$$

Hence

$$\mu_{p:2p}^2 - \mu_{p+1:2p}^2 = -\theta^2 \left[\frac{2}{p} \sum_{j=p+1}^{2p} \frac{1}{j} + \frac{2}{p^2} \right] \quad (18)$$

$$= -\theta^2 S \quad (19)$$

where

$$S = \frac{2}{p} \sum_{j=p+1}^{2p} \frac{1}{j} + \frac{2}{p^2}. \quad (20)$$

Using (15), (16) ,(17) and (19), mean and variance of $Y_{[r:n]}$ may be obtained as,

$$\begin{aligned} \mu_{[r:n]} &= \theta_2 \left(1 - \frac{A}{p} \right) \\ &= \theta_2 \left(1 - \alpha m \frac{B(m+r-1, m+n-r)}{B(r, n-r+1)} \cdot \frac{n-2r+1}{2m+n-1} \cdot B(p, p+1) \right) \end{aligned} \quad (21)$$

and

$$V(Y_{[r:n]}) = \theta_2^2 \left(1 + \frac{2A}{p} - \frac{A^2}{p^2} - AS \right). \quad (22)$$

From Thomas (2015), the joint c.d.f of $(Y_{[r:n]}, Y_{[s:n]})$ is

$$\begin{aligned} F_{[r,s,n]}(y_1, y_2) &= \left(1 - \exp\left(\frac{-y_1}{\theta_2}\right) \right) \left(1 - \exp\left(\frac{-y_2}{\theta_2}\right) \right) \\ &\quad + \gamma_1 \left(1 - \exp\left(\frac{-y_1}{\theta_2}\right) \right) \left[\left(1 - \exp\left(\frac{-y_2}{\theta_2}\right) \right) \exp\left(\frac{-y_2}{\theta_2}\right) \right]^p \\ &\quad + \gamma_2 \left(1 - \exp\left(\frac{-y_2}{\theta_2}\right) \right) \left[\left(1 - \exp\left(\frac{-y_1}{\theta_2}\right) \right) \exp\left(\frac{-y_1}{\theta_2}\right) \right]^p \\ &\quad + \gamma_3 \left[\left(1 - \exp\left(\frac{-y_1}{\theta_2}\right) \right) \exp\left(\frac{-y_1}{\theta_2}\right) \right]^p \\ &\quad \times \left[\left(1 - \exp\left(\frac{-y_2}{\theta_2}\right) \right) \exp\left(\frac{-y_2}{\theta_2}\right) \right]^p, \\ &\quad 0 < y_1, y_2 < \infty \end{aligned} \quad (23)$$

where

$$\gamma_1 = \alpha m \cdot \frac{1}{B(r, s-r, n-s+1)} \\ \times \left\{ \sum_{j=0}^{m-1} \binom{m-1}{j} (-1)^j B(r, s-r, m+n+j-s) \left(1 - \frac{2s}{m+n+j} \right) \right\}$$

$$\gamma_2 = \alpha m \cdot \frac{1}{B(r, s-r, n-s+1)} \\ \times \left\{ \sum_{j=0}^{m-1} \binom{m-1}{j} (-1)^j B(m+j+r-1, s-r, n-s+1) \left(1 - \frac{2r}{m+n+j} \right) \right\}$$

and

$$\gamma_3 = \alpha^2 m^2 \cdot \frac{1}{B(r, s-r, n-s+1)} \\ \times \left[\sum_{j,k=0}^{m-1} \binom{m-1}{j} (-1)^j \binom{m-1}{k} (-1)^k B(m+k+r-1, s-r, m+n+j-s) \right. \\ - 2 \sum_{j,k=0}^{m-1} \binom{m-1}{j} (-1)^j \binom{m-1}{k} (-1)^k B(m+k+r, s-r, m+n+j-s) \\ - 2 \sum_{j,k=0}^{m-1} \binom{m}{j} (-1)^j \binom{m-1}{k} (-1)^k B(m+k+r-1, s-r, m+n+j-s) \\ \left. + 4 \sum_{j,k=0}^{m-1} \binom{m}{j} (-1)^j \binom{m-1}{k} (-1)^k B(m+k+r, s-r, m+n+j-s) \right].$$

Joint probability density function corresponding to (23) is

$$\begin{aligned}
 f_{[r,s;n]}(y_1, y_2) &= \frac{1}{\theta_2^2} \exp\left(-\frac{y_1 + y_2}{\theta_2}\right) \\
 &\times \left[1 + \gamma_1 p \left\{ 2 \exp\left(-\frac{y_2}{\theta_2}\right) - 1 \right\} \left\{ \left(1 - \exp\left(-\frac{y_2}{\theta_2}\right) \right) \exp\left(-\frac{y_2}{\theta_2}\right) \right\}^{p-1} \right. \\
 &+ \gamma_2 p \left\{ 2 \exp\left(-\frac{y_1}{\theta_2}\right) - 1 \right\} \left\{ \left(1 - \exp\left(-\frac{y_1}{\theta_2}\right) \right) \exp\left(-\frac{y_1}{\theta_2}\right) \right\}^{p-1} \\
 &+ \gamma_3 p^2 \left\{ 2 \exp\left(-\frac{y_1}{\theta_2}\right) - 1 \right\} \left\{ 2 \exp\left(-\frac{y_2}{\theta_2}\right) - 1 \right\} \\
 &\times \left. \left\{ \left(1 - \exp\left(-\frac{y_1}{\theta_2}\right) \right) \exp\left(-\frac{y_1}{\theta_2}\right) \right\}^{p-1} \left\{ \left(1 - \exp\left(-\frac{y_2}{\theta_2}\right) \right) \exp\left(-\frac{y_2}{\theta_2}\right) \right\}^{p-1} \right] \\
 &0 < y_1, y_2 < \infty. \tag{24}
 \end{aligned}$$

Writing $1-2F = 1-F-F$ and using the representation of order statistics we write (24) as

$$\begin{aligned}
 f_{[r,s;n]}(y_1, y_2) &= f_Y(y_1) f_Y(y_2) \\
 &+ \gamma_1 p f_Y(y_1) \{B(p, p+1) f_{p:2p}(y_2) - B(p, p+1) f_{p+1:2p}(y_2)\} \\
 &+ \gamma_2 p f_Y(y_2) \{B(p, p+1) f_{p:2p}(y_1) - B(p, p+1) f_{p+1:2p}(y_1)\} \\
 &+ \gamma_3 p^2 \{B(p, p+1) f_{p:2p}(y_1) - B(p, p+1) f_{p+1:2p}(y_1)\} \\
 &\times \{B(p, p+1) f_{p:2p}(y_2) - B(p, p+1) f_{p+1:2p}(y_2)\}. \tag{25}
 \end{aligned}$$

where

$$\begin{aligned}
 f_{p:2p}(y) &= \frac{(2p)!}{(p-1)! p!} [F(y)]^{p-1} [1 - F(y)]^p f(y) \\
 &= \frac{(2p)!}{(p-1)! p!} \theta_2 \left(1 - \exp\left(-\frac{y}{\theta_2}\right) \right)^{p-1} \exp\left(-\frac{(p+1)y}{\theta_2}\right), \quad y \geq 0
 \end{aligned}$$

and

$$\begin{aligned}
 f_{p+1:2p}(y) &= \frac{(2p)!}{p!(p-1)!} [F(y)]^p [1 - F(y)]^{p-1} f(y) \\
 &= \frac{(2p)!}{p!(p-1)!} \theta_2 \left(1 - \exp\left(-\frac{y}{\theta_2}\right) \right)^p \exp\left(-\frac{py}{\theta_2}\right), \quad y \geq 0.
 \end{aligned}$$

Product moments for $l, k > 0$ are given by

$$\begin{aligned}
 \mu_{[r,s:n]}(l, k) &= E(Y_{[r:n]}^l Y_{[s:n]}^k) \\
 &= \mu^{(l)} \mu^{(k)} + \gamma_1 p \left[B(p, p+1) \mu_{p:2p}^{(k)} - B(p, p+1) \mu_{p+1:2p}^{(k)} \right] \mu^{(l)} \\
 &\quad + \gamma_2 p \left[B(p, p+1) \mu_{p:2p}^{(l)} - B(p, p+1) \mu_{p+1:2p}^{(l)} \right] \mu^{(k)} \\
 &\quad + \gamma_3 p^2 \left[B(p, p+1) \mu_{p:2p}^{(l)} - B(p, p+1) \mu_{p+1:2p}^{(l)} \right] \left[B(p, p+1) \mu_{p:2p}^{(k)} \right. \\
 &\quad \left. - B(p, p+1) \mu_{p+1:2p}^{(k)} \right]. \tag{26}
 \end{aligned}$$

Hence from (17) and (26), we obtain

$$\begin{aligned}
 \mu_{[r,s:n]} &= \theta_2^2 - \theta_2^2 \gamma_1 B(p, p+1) \\
 &\quad - \theta_2^2 \gamma_2 B(p, p+1) \\
 &\quad + \theta_2^2 \gamma_3 [B(p, p+1)]^2. \tag{27}
 \end{aligned}$$

Using (21) and (27) it follows that

$$\begin{aligned}
\text{Cov}(Y_{[r:n]}, Y_{[s:n]}) &= \mu_{[r,s:n]} - \mu_{[r:n]} \mu_{[s:n]} \\
&= \theta_2^2 - \theta_2^2 \gamma_1 B(p, p+1) \\
&\quad - \theta_2^2 \gamma_2 B(p, p+1) \\
&\quad + \theta_2^2 \gamma_3 [B(p, p+1)]^2 \\
&\quad - \theta_2 \left(1 - \alpha m \frac{B(m+r-1, m+n-r)}{B(r, n-r+1)} \frac{n-2r+1}{2m+n-1} B(p, p+1) \right) \\
&\quad \times \theta_2 \left(1 - \alpha m \frac{B(m+s-1, m+n-s)}{B(s, n-s+1)} \frac{n-2s+1}{2m+n-1} B(p, p+1) \right) \\
&= \theta_2^2 B(p, p+1) \left(\alpha m \frac{B(m+r-1, m+n-r)}{B(r, n-r+1)} \cdot \frac{n-2r+1}{2m+n-1} \right. \\
&\quad + \alpha m \frac{B(m+r-1, m+n-r)}{B(s, n-r+1)} \cdot \frac{n-2r+1}{2m+n-1} \\
&\quad - \gamma_1 - \gamma_2 + \gamma_3 B(p, p+1) \\
&\quad - (\alpha m)^2 \frac{B(m+r-1, m+n-r)}{B(r, n-r+1)} \cdot \frac{n-2r+1}{2m+n-1} \\
&\quad \times \left. \frac{B(m+s-1, m+n-s)}{B(s, n-s+1)} \cdot \frac{n-2s+1}{2m+n-1} B(p, p+1) \right) \quad (28)
\end{aligned}$$

The overall covariance increases with θ_2 . So larger dispersion in the Y -marginal amplifies dependence between concomitants $Y_{[r:n]}$ and $Y_{[s:n]}$. For positive α , the covariance tends to be positive and increases as r and s approach each other.

We propose a quick estimator for θ_2 derived from the quasi-mid-range, defined as

$$T_r = \frac{Y_{[r:n]} + Y_{[n-r+1:n]}}{2}, \quad r = 1, 2, \dots, \left\lfloor \frac{n+1}{2} \right\rfloor.$$

From (24) we get

$$\begin{aligned}
 E[T_r] &= E\left[\frac{Y_{[r:n]} + Y_{[n-r+1:n]}}{2}\right] \\
 &= \frac{1}{2} \left[\theta_2 \left(1 - \alpha m \frac{B(m+r-1, m+n-r)}{B(r, n-r+1)} \cdot \frac{n-2r+1}{2m+n-1} B(p, p+1) \right) \right. \\
 &\quad \left. + \theta_2 \left(1 + \alpha m \frac{B(m+r-1, m+n-r)}{B(r, n-r+1)} \cdot \frac{n-2r+1}{2m+n-1} B(p, p+1) \right) \right] \\
 &= \theta_2.
 \end{aligned} \tag{29}$$

Thus, all quartile midranges are unbiased estimators of the location parameter.

Using (22) and (28), the variance of T_r is

$$\begin{aligned}
 V(T_r) &= \frac{1}{4} \left(V(Y_{[r:n]}) + V(Y_{[n-r+1:n]}) + 2 \operatorname{Cov}(Y_{[r:n]}, Y_{[n-r+1:n]}) \right) \\
 &= \frac{1}{2} \theta_2^2 \left[1 - \frac{\alpha m B(p+1, p)}{B(r, n-2r+1, r)} \left\{ \sum_{j=0}^{m-1} \binom{m-1}{j} (-1)^j B(r, n-2r+1, m+j+r-1) \right. \right. \\
 &\quad \times \left(1 - \frac{2(n-r+1)}{m+n+j} \right) + \sum_{j=0}^{m-1} \binom{m-1}{j} (-1)^j B(m+j+r-1, n-2r+1, r) \\
 &\quad \times \left(1 - \frac{2r}{m+n+j} \right) \left. \right\} \\
 &\quad + \frac{[\alpha m B(p+1, p)]^2}{B(r, n-2r+1, r)} \left\{ \sum_{j,k=0}^{m-1} (-1)^{j+k} \binom{m-1}{j} \binom{m-1}{k} \right. \\
 &\quad \times B(m+k+r-1, n-2r+1, m+j+r-1) \\
 &\quad - 2 \sum_{j,k=0}^{m-1} (-1)^{j+k} \binom{m-1}{j} \binom{m-1}{k} B(m+k+r, n-2r+1, m+j+r-1) \\
 &\quad - 2 \sum_{j,k=0}^{m-1} (-1)^{j+k} \binom{m}{j} \binom{m-1}{k} B(m+k+r-1, n-2r+1, m+j+r-1) \\
 &\quad \left. \left. + 4 \sum_{j,k=0}^{m-1} (-1)^{j+k} \binom{m}{j} \binom{m-1}{k} B(m+k+r, n-2r+1, m+j+r-1) \right\} \right].
 \end{aligned} \tag{30}$$

The variance depends explicitly on sample size n , choice r , and dependence parameter α . Stronger dependence inflates or reduces variance depending on the sign of α . Shape parameters m and p control the distributional form and hence the efficiency of the estimator. The quartile midrange is less sensitive to extreme values than the full midrange, making it particularly useful for heavy-tailed or skewed exponential-type data. It is especially attractive in symmetric or weakly dependent settings. The variance is minimized when the pair is as central as possible. By symmetry of the order indices, this occurs at the median ranks.

That is, the median concomitant

$$\begin{aligned} T_{\frac{n+1}{2}} &= \frac{Y_{[\frac{n+1}{2}:n]} + Y_{[n-\frac{n+1}{2}+1:n]}}{2} \\ &= Y_{[\frac{n+1}{2}:n]} \end{aligned}$$

is the statistic that minimizes variance.

References

- [1] Ahsanullah, M. (1994). Record values, random record models and concomitants. *Journal of Statistical Research*, 28, 89-109.
- [2] Bhattacharya, P.K. (1974). Convergence of sample paths of normalized sums of induced order statistics. *Annals of Statistics*, 2, 1034-1039.
- [3] Bairamov, I.G. and Kotz, S. (2000a). Dependence Structure and Symmetry of Huang-Kotz FGM Distributions and their Extensions. Technical Report, The George Washington University.
- [4] Bairamov, I., Kotz, S. and Bekci, M. (2001). New generalized Farlie-Gumbel-Morgenstern distributions and concomitants of order statistics. *Journal of Applied Statistics*, 28(5), 521-536.
- [5] Balasubramanian, K. and Beg, M.I. (1997). Concomitants of order statistics in Morgenstern type bivariate exponential distribution. *Journal of Applied Statistics*, 5, 233-245.
- [6] Barakat, H. M., Alawady, M. A., Taher, T. S. and Hussein, I. A. (2025). Second-order concomitants of order statistics from bivariate Cambanis family: some information measures—estimation. *Communications in Statistics -*

Simulation and Computation, 54(8), 3195-3221.

DOI: <https://doi.org/10.1080/03610918.2024.2344023>

- [7] Beg, M. I. and Ahsanullah, M. (2008). Concomitants of generalized order statistics from Farlie-Gumbel-Morgenstern distributions. *Statistical Methodology*, 5, 1-20.
- [8] David, H.A. (1973). Concomitants of order statistics. *Bulletin of the International Statistical Institute*, 45, 295-300.
- [9] David, H.A. and Galambos, J. (1974). The asymptotic theory of concomitants of order statistics. *Journal of Applied Probability*, 11, 762-770.
- [10] D'este, G.M. (1981). A Morgenstern-type gamma distribution. *Biometrika*, 68(1), 339-340.
- [11] Gumbel, E.J. (1960). Bivariate exponential distribution. *Journal of the American Statistical Association*, 55, 698-707.
- [12] Gumbel, E.J. (1961). Bivariate logistic distribution. *Journal of the American Statistical Association*, 56, 335-349.
- [13] Huang, J.S. and Kotz, S. (1999). Modifications of the Farlie-Gumbel-Morgenstern distributions. A tough hill to climb. *Metrika*, 49, 307-323.
- [14] Johnson, N.L. and Kotz, S. (1975). On some generalized Farlie-Gumbel-Morgenstern distributions. *Communications in Statistics – Theory and Methods*, 4, 415-427.
- [15] Mardia, K.V. (1970). Families of Bivariate Distributions. Charles Griffin and Company, London.
- [16] Morgenstern, D. (1956). Einfache Beispiele zweidimensionaler Verteilungen. *Mitteilungsblatt für Mathematische Statistik*, 8, 234-235.
- [17] Nair, N.U. and Scaria, J. (2010). General properties and concomitants of Morgenstern bivariate gamma distribution and their applications in estimation. *Statistics*, 44(2), 155-167.
- [18] Nelson, R.B. (2006). An Introduction to Copulas (2nd ed.). Springer, New York.

- [19] Scaria, J. and Nair, N.U. (1999). On concomitants of order statistics from Morgenstern family. *Biometrical Journal*, 41(4), 483-489.
- [20] Scaria, J. and Nair, N.U. (2005). Distribution of concomitants of selected order statistics from Morgenstern family. *Journal of Applied Statistical Science*, 14(3), 251-262.
- [21] Scaria, J. and Nair, N.U. (2008). Distribution of the extremes of concomitants from Morgenstern family. *Statistical Papers*, 49(1), 109-119.
- [22] Scaria, J. and Thomas, B. (2008). Concomitants of order statistics from Generalized Morgenstern family. *STARS: International Journal*, 2(2), 153-162.
- [23] Scaria, J. and Thomas, B. (2013). Second order concomitants from the Morgenstern family of distributions. *Journal of Applied Statistical Science*, 21(1), 63-76.
- [24] Thomas, B. and Scaria, J. (2011). Concomitants of order statistics from bivariate Cambanis family. *Science & Society*, 9(1), 49-62.
- [25] Thomas, B. (2015). Concomitants of Order Statistics from Bivariate Generalized Morgenstern and Cambanis Family. Ph.D thesis, Department of Statistics, Nirmala College, Muvattupuzha.
- [26] Thomas, P.Y. and Veena, T.G. (2014). Characterizations of bivariate distributions using concomitants of record values. *Metrika*, 77, 947-963.
- [27] Yang, S.S. (1977). General distribution theory of the concomitants of order statistics. *Annals of Statistics*, 5, 996-1002.
- [28] Yeo, W.B. and David, H.A. (1984). Selection through an associated characteristic with applications to random effects model. *Journal of the American Statistical Association*, 79, 399.
- [29] Veena, T.G. and Thomas, P.Y. (2008). Applications of concomitants of order statistics in characterizing bivariate distributions. *Statistics Probability Letters*, 78, 3350-3354.