



Research Article

A comparison of Exponentiated Exponential (EE) distribution and Generalised Beta distribution of Second kind (GB2)

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Abstract

This study presents a comprehensive comparison of Exponentiated Exponential(EE) distribution and Generalised Beta distribution of second kind(GB2) based on their properties, parameter estimation techniques, and goodness-of-fit measures. The both Exponentiated Exponential and Generalised Beta distribution of second kind are life time distributions and widely used in statistical modeling due to their flexibility in capturing various data behaviors used to fit skewed, positive data such as income, reliability, and survival times. But they differ in complexity and tail behavior. The important statistical characteristics including moments, hazard functions, fitted values, and entropy measurements were discussed. Parameters are estimated using Bayesian estimation techniques and Maximum Likelihood Estimation (MLE), AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion) and Shannon differential entropy.

Keywords: Exponentiated Exponential distribution, Generalised Beta distribution of second kind, Maximum Likelihood Estimator(MLE), Hazard functions, Akaike Information Criterion(AIC), Bayesian Information Criterion(BIC), Shannon differential entropy, Survival functions.

1 Introduction

The statistical distribution theory deals with the study of probability distributions and how they are applied to real life scenarios. It helps to provide the mathematical model to variability of data. The distributions may be discrete or continuous, depending on the type of data. Most of these distributions are already discussed in literature. Features of random variables such as the probability mass function (pmf) and probability density function (pdf), cumulative distribution function (cdf), moments, parameter estimation, data fitting, hazard functions, survival functions, tail behaviour and distribution flexibility are also covered by distribution theory. In this article, we compare and analyze the Exponentiated Exponential distribution and Generalised Beta distribution of second kind[3],[11].

The Exponentiated Exponential distribution has been proposed by the authors by two parameters recently. Gupta and Kundu have done substantial research on it[6],[7],[8],[2],[14]. The key difference between the Exponentiated Exponential distribution and the Generalised Beta distribution of the second kind is that the exponential distribution primarily models, time between events with a constant failure rate, while GB2 distribution is more flexible capable of capturing both positive and negative skewness. It is often used to model data with skewed or heavy-tailed distributions such as income or flood frequency, where the range is typically bounded on one side (positive values only).

A probability distribution with four parameters is called the Generalised Beta distribution of second Kind. It guarantees the features of dispersion metrics like skewness and kurtosis, including low, middle, and right-hand tails. The GB2 distribution includes several other distributions as special or limiting instances, such as generalised gamma (GG), Dagum, beta of the second type (B2), Singh-Maddala (SMI), gamma, Weibull, and exponential distribution. It has been studied in literature [5],[9],[10],[12],[4],[15].

In general we can say that the Exponentiated Exponential distribution is best suited for modeling constant failure rate scenarios, while the Generalised Beta distribution of the second kind is more versatile for modeling data with varying skewness and bounded ranges. This article is explained as follows : It explains the probability density function, fitted distribution, moments and parameter estimation of EE and GB2 distributions. Additionally, real-world and simulated

data are used to compare these two distributions using AIC (Akaike Information Criterion) and BIC (Bayesian Information Criterion)[1].

2 Mathematical properties and behaviours

2.1 Probability density functions and its moments

2.1.1 The pdf of Exponentiated Exponential distribution with parameters $\hat{\alpha}$ and $\hat{\lambda}$

The probability density function (pdf) of a random variable Y with an exponentiated exponential distribution is as follows;

$$f_{EE}(Y, \hat{\alpha}, \hat{\lambda}) = \hat{\alpha}\hat{\lambda}e^{-\hat{\lambda}y}(1 - e^{-\hat{\lambda}y})^{\hat{\alpha}-1} \quad ; y, \hat{\alpha}, \hat{\lambda} > 0 \quad (1)$$

Where $\hat{\alpha}$ is the shape parameter and $\hat{\lambda}$ is the scale parameter.

For different values of $\hat{\alpha}$, the authors provided the graphs of the EE distribution.

The EE's probability density function (pdf) can have a variety of forms.

For $\hat{\alpha} \leq 1$, it is an increasing function and for $\hat{\alpha} \geq 1$, It resembles the Weibull and gamma distributions. Also it is unimodal, skewed, right-tailed and it is not symmetric for any large shape parameter.

For $\hat{\alpha} > 1$, the mode is at $\log \hat{\alpha}$ and for $\hat{\alpha} \leq 1$, the mode is at 0.

The moment generating function may be used to produce the various moments of the exponentiated exponential distribution.

If the random variable Y follows $EE(\hat{\alpha}, \hat{\lambda})$, then the moment generating function, $M(t)$ of Y for $t < \hat{\lambda}$ is

$$M(t) = E(e^{ty}) = \frac{\Gamma(\hat{\alpha} + 1)\Gamma(1 - \frac{t}{\hat{\lambda}})}{\Gamma(\hat{\alpha} - \frac{t}{\hat{\lambda}} + 1)} \quad (2)$$

If the random variable Y has the pdf as in (1) with $\hat{\lambda} = 1$, then the moment generating function $M(t)$, of Y for $|t| < 1$ is

$$M(t) = E(e^{ty}) = \frac{\Gamma(\hat{\alpha} + 1)\Gamma(1 - t)}{\Gamma(\hat{\alpha} - t + 1)} \quad (3)$$

From this, we obtained,

$$Mean = E(Y) = \frac{1}{\hat{\lambda}}[\psi(\hat{\alpha} + 1) - \psi(1)]$$

$$Variance = V(Y) = \frac{1}{\hat{\lambda}^2}[\psi'(1) - \psi'(\hat{\alpha} + 1)]$$

where, $\psi(.)$ and $\psi'(.)$ are digamma and polygamma function respectively.

2.1.2 The pdf of Generalised Beta of second kind with parameters $\hat{a}, \hat{b}, \hat{p}$ and $\hat{q}$

The probability density function is of the form,

$$f(x) = \frac{\hat{a}x^{\hat{a}\hat{p}-1}}{\hat{b}^{\hat{a}\hat{p}}B(\hat{p}, \hat{q})[1 + (\frac{x}{\hat{b}})^{\hat{a}}]^{\hat{p}+\hat{q}}} \quad ; x > 0 \text{ and } \hat{a}, \hat{b}, \hat{p}, \hat{q} > 0 \quad (4)$$

where $\hat{b}$ is the scale parameter and $\hat{a}, \hat{p}, \hat{q}$ are shape parameters.

$$B(\hat{p}, \hat{q}) = \frac{\Gamma\hat{p}\Gamma\hat{q}}{\Gamma(\hat{p} + \hat{q})} \text{ is the beta function.}$$

$\Gamma.$ is the gamma function.

The k^{th} moment of the GB2 distribution is given by

$$E(X^k) = \frac{\hat{b}^k \Gamma(\hat{p} + \frac{k}{\hat{a}}) \Gamma(\hat{q} - \frac{k}{\hat{a}})}{\Gamma\hat{p}\Gamma\hat{q}} \quad \text{and exist only if } -\hat{a}\hat{p} < k < \hat{a}\hat{q}$$

where $\hat{a}\hat{p}$ and $\hat{a}\hat{q}$ are lower and upper tails respectively. These tails depend on the distribution; a higher value of $\hat{a}$ reduces the pdf at both tails, and skewness is affected by the relative sizes of $\hat{p}$ and $\hat{q}$.

From this, we obtained,

$$Mean = E(X) = \hat{b} \frac{\Gamma(\hat{p} + \frac{1}{\hat{a}}) \Gamma(\hat{q} - \frac{1}{\hat{a}})}{\Gamma(\hat{p}) \Gamma(\hat{q})}$$

$$Variance = V(X) = \hat{b}^2 \frac{\Gamma(\hat{p} + \frac{2}{\hat{a}}) \Gamma(\hat{q} - \frac{2}{\hat{a}})}{\Gamma(\hat{p}) \Gamma(\hat{q})} - (\hat{b} \frac{\Gamma(\hat{p} + \frac{1}{\hat{a}}) \Gamma(\hat{q} - \frac{1}{\hat{a}})}{\Gamma(\hat{p}) \Gamma(\hat{q})})^2$$

2.2 Measuring the dispersion

Digamma and polygamma functions may be used to express the second and third central moments of the EE distribution as

$$\mu_2 = \frac{1}{\hat{\lambda}^2} [\psi'(1) - \psi'(\hat{\alpha} + 1) + (\psi(\hat{\alpha} + 1) - \psi(1))^2] \quad (5)$$

$$\mu_3 = \frac{1}{\hat{\lambda}^3} [\psi''(\hat{\alpha} + 1) - \psi''(1) + 3(\psi(\hat{\alpha} + 1) - \psi(1)) - (\psi'(1) - \psi'(\hat{\alpha} + 1)) + (\psi(\hat{\alpha} + 1) - \psi(1))^3] \quad (6)$$

The Generalised Beta of second kind distribution with parameters $\hat{a}, \hat{b}, \hat{p}$ and $\hat{q}$ has the skewness (γ_1) as follows.

$$\gamma_1(GB2) = \frac{2(\hat{q} - \hat{p})\sqrt{\hat{p} + \hat{q} - 1}}{(\hat{q} - 2)\sqrt{\hat{p}(\hat{q} - 3)}} \quad (7)$$

The parameter $\hat{b}$ (scale parameter) does not affect skewness, as it only shifts the distribution. The skewness is exist if $\hat{q} > 3$.

2.3 Estimation of parameters

For Exponentiated Exponential and Generalised Beta distribution of Second kind, the parameters are estimated by the method of MLE. There are several methods are available in literature [13]. The parameters of EE distribution can also be estimated by the method of moment estimators, percentile estimators, least square estimators.

If $X_1, X_2, X_3, \dots, X_n$ are random samples from (1), then α, λ represents the estimated values of the parameters $\hat{\alpha}, \hat{\lambda}$ of the EE distribution. MLE uses this estimating technique, it is possible to acquire λ as the fixed point solution of

$$h(\hat{\lambda}) = \lambda,$$

where,

$$h(\hat{\lambda}) = \left[\frac{\sum_{i=1}^n \left(\frac{x_i e^{-\hat{\lambda} x_i}}{1 - e^{-\hat{\lambda} x_i}} \right)}{\sum_{i=1}^n \ln(1 - e^{-\hat{\lambda} x_i})} + \frac{1}{n} \sum_{i=1}^n \frac{x_i}{1 - e^{-\hat{\lambda} x_i}} \right]^{-1}$$

Once we obtained the estimated value λ , the α can be obtained as

$$\alpha = -\frac{n}{\sum_{i=1}^n \ln(1 - e^{-x_i \lambda})}$$

While the MLE approach is used to estimate the parameters of the GB2 distribution, iteration and the estimating procedure is continued using the Newton-Raphson approach. It has been studied in literature [16]. When the sample size rises, the estimation of parameters $\hat{a}$, $\hat{b}$, $\hat{p}$, $\hat{q}$ achieves values that are near to the actual values of the parameters. A greater sample size results in a narrower confidence interval (CI), as well as reduced bias and variation.

2.4 Survival and Hazard Functions

2.4.1 Survival Function of EE distribution and GB2 distribution

Survival function of EE distribution: The survival function, $S(x)$, is defined as $S(x) = P(X > x)$, if X is the random variable and x is a specific value of X . The survival function is derived from the cumulative distribution function (cdf). The cdf of EE distribution is:

$$F(x; \hat{\alpha}, \hat{\lambda}) = (1 - e^{-\hat{\lambda}x})^{\hat{\alpha}} ; x > 0; \hat{\alpha}, \hat{\lambda} > 0 \quad (8)$$

Where $\hat{\alpha}$ is the shape parameter and $\hat{\lambda}$ is the scale parameter. The survival function of EE distribution is given by

$$\begin{aligned} S(x) &= (1 - F(x)) \\ S(x) &= 1 - (1 - e^{-\hat{\lambda}x})^{\hat{\alpha}} \end{aligned} \quad (9)$$

Survival function of GB2 distribution: The cumulative distribution function(cdf) of GB2 distribution is given by

$$F(x; \hat{a}, \hat{b}, \hat{p}, \hat{q}) = I_v(\hat{p}, \hat{q}) , \quad x > 0 \quad (10)$$

Where $I_v(\hat{p}, \hat{q})$ is the regularized incomplete Beta function and,

$$v = \frac{\left(\frac{x}{b}\right)^{\hat{a}}}{1 + \left(\frac{x}{b}\right)^{\hat{a}}}$$

is the transformation applied to x .

where, $\hat{a} > 0$, is the shape parameter and $\hat{b} > 0$, is the scale parameter.

$\hat{p}, \hat{q} > 0$ are the Beta function parameters.

The survival function of GB2 distribution is given by

$$\begin{aligned} S(x) &= (1 - F(x)) \\ S(x) &= 1 - I_v(\hat{p}, \hat{q}) \end{aligned} \quad (11)$$

$$\text{Where, } v = \frac{(\frac{x}{\hat{b}})^{\hat{a}}}{1 + (\frac{x}{\hat{b}})^{\hat{a}}}$$

2.4.2 Hazard functions of EE and GB2 distribution

Hazard function of Exponentiated Exponential distribution: Let X is the random variable and $\hat{\alpha}$ and $\hat{\lambda}$ are the parameters, then the Hazard function of EE is defined by

$$H_{EE}(x) = \frac{f(x)}{S(x)} \quad (12)$$

where $f(x)$ and $S(x)$ represent the probability density function and survival function of the EE distribution respectively.

Where,

$$f(x) = \hat{\alpha} \hat{\lambda} e^{-\hat{\lambda}x} (1 - e^{-\hat{\lambda}x})^{\hat{\alpha}-1}$$

and

$$S(x) = (1 - e^{-\hat{\lambda}x})^{\hat{\alpha}}$$

then the Hazard function of EE distribution is given by

$$H_{EE}(x) = \frac{\hat{\alpha} \hat{\lambda} e^{-\hat{\lambda}x} (1 - e^{-\hat{\lambda}x})^{\hat{\alpha}-1}}{(1 - e^{-\hat{\lambda}x})^{\hat{\alpha}}}$$

by simplifying this we can get ,

$$H_{EE}(x) = \hat{\alpha} \hat{\lambda} e^{-\hat{\lambda}x} (1 - e^{-\hat{\lambda}x})^{-1} \quad (13)$$

Hazard function of GB2 distribution: Suppose X is the random variable and $\hat{a}, \hat{b}, \hat{p}, \hat{q}$ are the parameters of GB2 distribution, then the Hazard function of GB2 distribution is given by

$$H_{GB2}(x) = \frac{f(x)}{S(x)}$$

Where $f(x)$ and $S(x)$ are the pdf and Survival function of GB2 distribution respectively.

Where

$$f(x) = \frac{\hat{a}x^{\hat{a}\hat{p}-1}}{\hat{b}^{\hat{a}\hat{p}}B(\hat{p},\hat{q})[1+(\frac{x}{\hat{b}})^{\hat{a}}]^{\hat{p}+\hat{q}}}, \quad (14)$$

$$S(x) = 1 - I_v(\hat{p}\hat{q})$$

Then, Hazard function of GB2 distribution is given by

$$H_{GB2}(x) = \frac{\frac{\hat{a}x^{\hat{a}\hat{p}-1}}{\hat{b}^{\hat{a}\hat{p}}B(\hat{p},\hat{q})[1+(\frac{x}{\hat{b}})^{\hat{a}}]^{\hat{p}+\hat{q}}}}{1 - I_v(\hat{p},\hat{q})} \quad (15)$$

where,

$$v = \frac{(\frac{x}{\hat{b}})^{\hat{a}}}{1 + (\frac{x}{\hat{b}})^{\hat{a}}}$$

3 Simulation Study, Real Data Analysis and Comparison

In this section, simulated data and one real life survival data set(Haberman's survival data set from kaggle) using Exponentiated Exponential and Generalized beta distribution of second kind are analyzed. At first, we fit both EE and GB2 distribution to the simulated data. Also analyzed and compared their performance using the metrics like AIC, BIC and entropy etc. We can analyze the performances using visually fit by plotting their pdf, cdf, survival function and hazard function.

Also, the both EE and GB2 distributions are applied to real life survival data. We can see many similarities by fitted distributions, AIC, BIC and Entropy plotting. The aim is to illustrate how the both EE and GB2 distributions can be applied and compared these two distributions to both simulated and real life data. All simulations, analysis and plots are done using programme codes written in R. For analyzing and comparing these EE distribution and GB2 distribution, we used

simulated data set and real life data (Haberman's survival data). The simulation for EE and GB2 distribution was done for different values of their parameters. The results are presented in Table 1. The real life data analysis for EE and GB2 distribution was done for different values of their parameters. The results are presented in Table 2.

Sl.No	EE Parameters ($\hat{\alpha}, \hat{\lambda}$)	Model	GB2 Parameters ($\hat{a}, \hat{b}, \hat{p}, \hat{q}$)	AIC	BIC	Skewness	Kurtosis
1	(1.5,0.5)	EE	-	710.5	715.2	2.31	8.12
	-	GB2	(2,1,1.5,2)	715.3	722.9	2.35	8.4
2	(2,0.3)	EE	-	685.8	690.1	1.89	6.23
	-	GB2	(2.5,1.5,1.8,2.2)	692.4	700	1.94	6.55
3	(0.8,1)	EE	-	740.7	745.4	3.12	11.1
	-	GB2	(1.5,0.8,1.2,1.6)	743.1	750.2	3.1	11.32

Table 1: Result of simulation study on EE and GB2 distribution

Sl.No	EE Parameters ($\hat{\alpha}, \hat{\lambda}$)	Model	GB2 Parameters ($\hat{a}, \hat{b}, \hat{p}, \hat{q}$)	AIC	BIC	Skewness	Kurtosis
1	(1.5,0.3)	EE	-	234.2	239.3	2.75	10.01
	-	GB2	(2,1,2,2.5,2)	230.5	240.6	2.68	9.87
2	(2,0.5)	EE	-	233.1	238	2.51	9.13
	-	GB2	(2.5,1,1.8,2.2)	231.9	241.1	2.49	9.02
3	(0.8,4)	EE	-	236.4	241.5	3.42	12.85
	-	GB2	(1.8,0.9,1.4,2)	229.7	239.9	3.31	12.1

Table 2: Result of Haberman's data analysis on EE and GB2 distribution.

The two-parameter ($\hat{\alpha}, \hat{\lambda}$) exponentiated exponential distribution, as opposed to the two-parameter gamma or two-parameter Weibull distribution that has already been explored in literature, is shown to be useful for analysing multiple lifetime data. By comparing EE and GB2 distribution, the AIC and BIC values of EE distribution for simulated data are smaller than that of GB2 distribution for different parameter values. Table 1 shows that EE distribution is best fit than GB2 distribution. But in the case of Haberman dataset, there is a slight variation. That is AIC and BIC values of GB2 are little more than EE distribution.

From the comparison result and plots, we can observe that both EE and GB2 distributions are statistically best fitted distribution especially analyzing the survival data. Since EE has two parameter distribution and GB2 has four parameter distribution, so that these differences in parameters count may affect the model fitting in the respective distributions. Therefore it may leads to the conclusion that GB2 have little more fit than EE distribution. That is more parameters may leads to more flexibility, skewed and the data has heavy tails.

However, for the simulated study and Haberman's data analysis, from the AIC

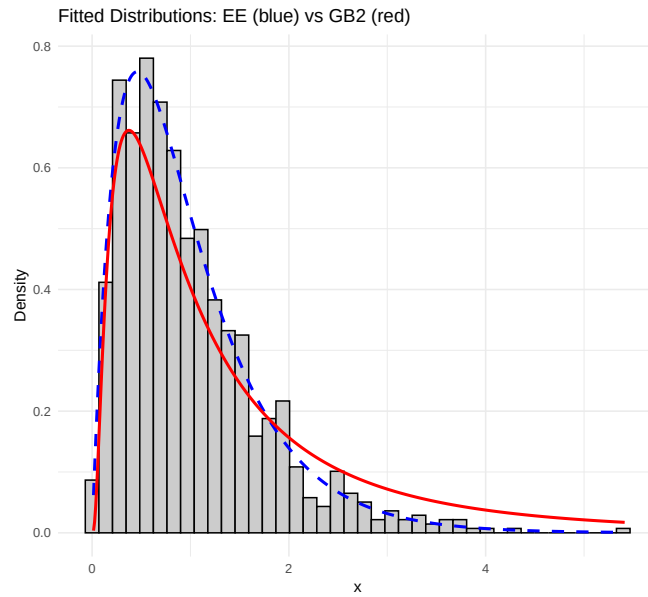


Figure 1: Fitted EE and GB2 distribution

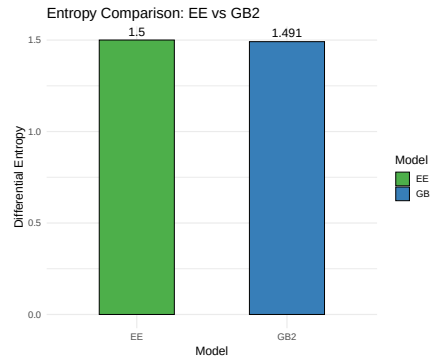


Figure 2: Entropy of EE and GB2 distribution

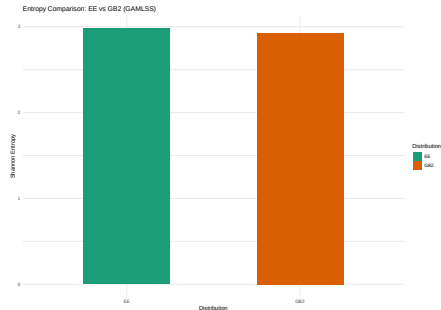


Figure 3: Entropy of EE and GB2 distribution for Haberman's data

and BIC, both the values are almost same. Also we can analyze that the entropy plot of both distributions for simulated and real life data set are similar in nature.

Generally we can conclude that two parameter EE distribution is the best fitted distribution like four parameter GB2 distribution.

4 Conclusion

This article proposed the comparison of Exponentiated Exponential distribution and Generalized Beta distribution of second kind. Both EE and GB2 distributions are widely used in statistical modeling due to their flexibility in capturing various data behaviours. The parameters of both models were estimated using both real and simulated data. The properties such as moments, hazard functions, fitted values, AIC, BIC and Shannon entropy measures are discussed in both simulated data and real life data. To estimate parameters of EE and GB2 distributions, we used the MLE method. From the model comparison of EE and GB2 distributions, we can observe that both the distributions are statistically best fitted distribution for analyzing the survival data. The EE is often used for modeling time until failure of machines, electronic components or medical devices, etc. Also used for medical survival analysis, that is used for modeling the survival time of patients after respective medical treatments. The GB2 distribution is often used in actuarial science to model loss severity in insurance claims. The EE distribution is compared with GB2 distribution, the EE provides more flexibility for complex survival rates.

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